GORENSTEIN INJECTIVE PRECOVERS, COVERS, AND ENVELOPES

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Abstract. We give a sufficient condition for the class of Gorenstein injective modules be precovering: if R is right noetherian and if the class of Gorenstein injective modules, $\mathcal{GI}$, is closed under filtrations, then $\mathcal{GI}$ is precovering in $R-Mod$. The converse is also true when we assume that $\mathcal{GI}$ is covering. We extend our results to the category of complexes. We prove that if the class of Gorenstein injective modules is closed under filtrations then the class of Gorenstein injective complexes is precovering in $Ch(R)$. We also give a sufficient condition for the existence of Gorenstein injective covers. We prove that if the ring R is commutative noetherian and such that the character modules of Gorenstein injective modules are Gorenstein flat, then the class of Gorenstein injective complexes is covering. And we prove that over such rings every complex also has a Gorenstein injective envelope. In particular this is the case when the ring is commutative noetherian with a dualizing complex. The second part of the paper deals with Gorenstein projective and flat complexes. We prove that over commutative noetherian rings of finite Krull dimension every complex of R-modules has a **special** Gorenstein projective precover.

1. Introduction

The Gorenstein injective modules were introduced by Enochs and Jenda (in [11]) as a generalization of the injective modules. Together with the Gorenstein projective and Gorenstein flat modules, they are the key ingredients of Gorenstein homological algebra. This is the reason why the existence of the Gorenstein resolutions has been studied extensively in the past years.

We consider the existence of Gorenstein injective precovers and covers. It is known that a precovering class that is closed under direct summands is also closed under arbitrary direct sums. So when a ring R is such that every R-module has a Gorenstein injective precover, every direct sum of Gorenstein injective modules is still Gorenstein injective.

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By [4] Proposition 3.15, such a ring R must be noetherian. It is not known whether the converse holds.
We give a sufficient condition for the class $\mathcal{GI}$ of Gorenstein injective modules be precovering: we prove that if R is right noetherian and if the class of Gorenstein injective modules is closed under filtrations, then $\mathcal{GI}$ is precovering in $R-Mod$. The converse is also true when we assume the existence of special Gorenstein injective precovers. In particular, this is the case when the class $\mathcal{GI}$ is covering.
We extend our results to the category $Ch(R)$ of complexes of R-modules over a two sided noetherian ring R. We prove that when $\mathcal{GI}$ is closed under filtrations the class of Gorenstein injective complexes is precovering.
We also show that when $\mathcal{GI}$ is closed under direct limits (and so $\mathcal{GI}$ is covering in $R-Mod$), the class of Gorenstein injective complexes is covering in $Ch(R)$. We prove that this is the case when R is commutative noetherian and such that for every Gorenstein injective module M, its character module M^+ is Gorenstein flat.
Then we consider the existence of Gorenstein injective envelopes for complexes. We prove (Theorem 5) that if R is a commutative noetherian ring with the property that the character modules of Gorenstein injectives are Gorenstein flat, then every complex has a Gorenstein injective envelope. In particular, this is the case when R is commutative noetherian with a dualizing complex.

As already noted, the Gorenstein injective, projective and flat modules are the key elements of Gorenstein homological algebra. Enochs and López-Ramos proved the existence of Gorenstein flat covers over coherent rings. And Jørgensen showed the existence of Gorenstein projective precovers over commutative noetherian rings with dualizing complexes. More recently, Murfet and Salarian extended his result to commutative noetherian rings of finite Krull dimension.
We extend some of the results on the existence of Gorenstein projective and Gorenstein flat (pre)covers to the category of complexes of R-modules over noetherian rings. We show the existence of Gorenstein flat covers over two sided noetherian rings. And we prove the existence of **special** Gorenstein projective precovers over commutative noetherian rings of finite Krull dimension. This apparently "slightly" variation is crucial from a homotopical point of view, since it allows to define Gorenstein projective cofibrant replacements of modules in the category of unbounded complexes.

2. Gorenstein injective precovers and covers for modules and for complexes

Throughout the paper R will denote an associative ring with 1. By R-module we mean left R-module.
We recall that a module G is Gorenstein injective if there is an exact and $Hom(Inj, -)$ exact complex $\ldots \to E_1 \to E_0 \to E_{-1} \to E_{-2} \to \ldots$ of injective modules such that $G = Ker(E_0 \to E_{-1})$.

The notions of precover and cover, preenvelope and envelope with respect to a class of modules $\mathcal{C}$ were introduced by Enochs in [7]. We are interested here in Gorenstein injective precovers and covers.

Definition 1. *A morphism $\phi : E \to X$ is a Gorenstein injective precover of X if E is Gorenstein injective and if $Hom(F, E) \to Hom(F, X)$ is surjective for all Gorenstein injective modules F. If moreover, any $f : E \to E$ such that $\phi \circ f = \phi$ is an automorphism of E then $\phi : E \to X$ is called a Gorenstein injective cover of X.*

Gorenstein injective preenvelopes and envelopes are defined dually.

Over Gorenstein rings the existence of Gorenstein injective covers is known (see for instance, [12, Theorem 11.1.3]). We give sufficient conditions in order for the existence of Gorenstein injective precovers and covers over noetherian rings.

We recall the following

Definition 2. *A direct system of modules $(X_\alpha | \alpha \leq \lambda)$ is said to be continuous if $X_0 = 0$ and if for each limit ordinal $\beta \leq \lambda$ we have $X_\beta = \varinjlim X_\alpha$ with the limit over the $\alpha < \beta$. The direct system $(X_\alpha | \alpha \leq \lambda)$ is said to be a system of monomorphisms if all the morphisms in the system are monomorphisms.*

If $(X_\alpha | \alpha \leq \lambda)$ is a continuous direct system of R-modules then for this to be a system of monomorphisms it suffices that $X_\alpha \to X_{\alpha+1}$ be monomorphism whenever $\alpha + 1 \leq \lambda$.

Definition 3. *Let $\mathcal{L}$ be a class of R-modules. An R-module X of A is said to be a direct transfinite extension of objects of $\mathcal{L}$ if $X = \varinjlim X_\alpha$ for a continuous direct system $(X_\alpha | \alpha \leq \lambda)$ of monomorphisms such that $coker(X_\alpha \to X_{\alpha+1})$ is in $\mathcal{L}$ whenever $\alpha + 1 \leq \lambda$.*

Definition 4. *By a filtration of a module M we mean that for some ordinal number λ we have a continuous well-ordered chain $(M_\alpha | \alpha \leq \lambda)$*

of submodules of M with $M_0 = 0$ and with $M_\lambda = M$. We say that λ is the length of the filtration. If $\mathcal{C}$ is any class of modules, this filtration is said to be a $\mathcal{C}$-filtration if for every $\alpha+1 \leq \lambda$ we have that $M_{\alpha+1}/M_\alpha$ is isomorphic to some $C \in \mathcal{C}$.
The class of all $\mathcal{C}$-filtered modules is denoted $Filt(\mathcal{C})$.

Roughly speaking, $Filt(\mathcal{C})$ is the class of all transfinite extensions of modules in $\mathcal{C}$.

It is known ([8], Theorem 5.5, and [26], Theorem in the Introduction) that if $\mathcal{C}$ is a **set** of modules, then $Filt(\mathcal{C})$ is precovering.

Our first result is a sufficient condition for the existence of Gorenstein injective precovers. It is known ([16]) that when R is right noetherian the class of Gorenstein injective left R-modules is a Kaplansky class. Since we use this property in our proof, we recall the definition:

Definition 5. *Let R be a ring, κ an infinite cardinal, and $\mathcal{A}$ a class of R-modules. $\mathcal{A}$ is κ-Kaplansky if for each $0 \neq A \in \mathcal{A}$ and each κ-generated submodule $B \subseteq A$ there exists a κ-presentable submodule $C \in \mathcal{A}$ such that $B \subseteq C \subseteq A$ and $A/C \in \mathcal{A}$.*

Lemma 1. *Let R be a right noetherian ring. There exists an infinite regular cardinal κ such that the class $\mathcal{GI}$ of Gorenstein injective left R-modules is a κ–Kaplansky class.*

Proof. This is [16, Proposition 2.6]. □

Proposition 1. *Let R be a right noetherian ring, κ an infinite regular cardinal as in Lemma 1, and let $\mathcal{X}$ denote a set of representatives of isomorphism classes of Gorenstein injective modules M such that $|M| \leq \kappa$. The following assertions are equivalent:*

(1) *$\mathcal{GI}$ is closed under $\mathcal{X}$–filtrations.*
(2) *$\mathcal{GI} = Filt(\mathcal{X})$.*

Proof. (2) $\Rightarrow$ (1) Clear.

(1) $\Rightarrow$ (2) By (1) it is clear that $Filt(\mathcal{X}) \subseteq \mathcal{GI}$. Conversely, let $G \neq 0$ be a Gorenstein injective module and let $\{g_\alpha, \alpha < \lambda\}$ be a generating set of G. Let $G_0 = 0$; if G_α is defined such that G_α is Gorenstein injective then let $A = G/G_\alpha$ and $B = (g_\alpha + G_\alpha)R$. Since $\mathcal{GI}$ is Kaplansky there exists $G_{\alpha+1} \subseteq G$ such that $G_\alpha \bigcup \{g_\alpha\} \subseteq G_{\alpha+1}$ and $G_{\alpha+1}/G_\alpha$ is

Gorenstein injective. Then $G/G_{\alpha+1} \simeq \frac{G/G_\alpha}{G_{\alpha+1}/G_\alpha}$ is Gorenstein injective and $G_{\alpha+1}$ is Gorenstein injective because $\mathcal{GI}$ is closed under extensions. If $\beta \leq \lambda$ is a limit ordinal, then we set $G_\beta = \bigcup_{\alpha<\beta} G_\alpha$. From (1) we infer that $G_\beta \in \mathcal{GI}$. Now, since the sequence $0 \to G_\beta \to G \to G/G_\beta \to 0$ is exact with G_β and G Gorenstein injectives, and $\mathcal{GI}$ is closed under cokernels of monomorphisms, it follows that G/G_β is Gorenstein injective, so we can continue the induction. The process clearly terminates. Thus $G \in Filt(\mathcal{X})$.

□

Theorem 1. *Under the assumptions of Proposition 1 the class of Gorenstein injective modules is precovering.*

Proof. By [8], Theorem 5.5, or by [26], Theorem in the Introduction, $Filt(\mathcal{X})$ is precovering. □

If moreover every R-module has a special $\mathcal{GI}$-precover then the converse is also true. As noted in the introduction, the class $\mathcal{GI}$ being precovering requires R to be left noetherian.

We show that in this case the class $\mathcal{GI}$ is closed under transfinite extensions.

Proposition 2. *Let R be a two sided noetherian ring. If every R-module has a special Gorenstein injective precover then the class $\mathcal{GI}$ of Gorenstein injective modules is closed under direct transfinite extensions.*

Proof. Let $(G_\alpha|\alpha \leq \lambda)$ be a direct system of monomorphisms, with each $G_\alpha \in \mathcal{GI}$, and let $G = \varinjlim G_\alpha$. Since for each α we have
$G_\alpha \in^\perp (\mathcal{GI}^\perp)$ it follows that $G = \varinjlim G_\alpha \in^\perp (\mathcal{GI}^\perp)$ (by [5], Theorem 1.2).
For each α consider $\oplus_{E\in X} E^{(Hom(E,G_\alpha))} \to G_\alpha$ where the map is the evaluation map, and X is a representative set of indecomposable injective modules E. This is an injective precover of G_α, and since G_α is Gorenstein injective, $\oplus_{E\in X} E^{(Hom(E,G_\alpha))} \to G_\alpha$ is surjective. Also this way of constructing a precover is functorial. The map $G_\alpha \to G_\beta$ gives rise to a a map $E_\alpha \to E_\beta$. Since $E_\alpha \to G_\alpha$ was constructed in a functorial manner, we have that when $\alpha \leq \beta \leq \gamma$ the map $E_\alpha \to E_\gamma$ is the composition of the two maps $E_\alpha \to E_\beta$ and $E_\beta \to E_\gamma$.

Then we have an exact sequence $E \to G \to 0$ with $E = \varinjlim E_\alpha$ an injective module. It follows that G has a surjective injective cover and therefore a surjective special Gorenstein injective precover. So there is an exact sequence

$$0 \to A \to \overline{G} \to G \to 0$$

with $A \in \mathcal{GI}^\perp$ and $\overline{G}$ Gorenstein injective. But by the above we have that $Ext^1(G, A) = 0$. So G is a direct summand of $\overline{G} \in \mathcal{GI}$. □

Corollary 1. *Let R be a two sided noetherian. If the class of Gorenstein injective modules, $\mathcal{GI}$, is covering then $\mathcal{GI}$ is closed under transfinite extensions.*

Proposition 3. *Let R be two sided noetherian. If every R-module has a special Gorenstein injective precover then $\mathcal{GI}$ is closed under $\mathcal{X}$-filtrations.*

Proof. By Proposition 2 above $Filt(\mathcal{X}) \subseteq \mathcal{GI}$. □

Proposition 4. *When the ring R is two sided noetherian and the class of Gorenstein injective modules is closed under direct limits, the class $\mathcal{GI}$ is covering.*

Proof. Since R is right noetherian the class of Gorenstein injective modules is Kaplansky. Since $\mathcal{GI}$ is also closed under direct limits, it is precovering (by Theorem 1). A precovering class that is also closed under direct limits is covering ([12], Corollary, 5.2.7) □

Corollary 2. *(*[12]*, Theorem 11.1.3) Over a Gorenstein ring the class of Gorenstein injective modules is covering.*

Corollary 3. *(*[22]*, Theorem 3.3) If R is commutative noetherian with a dualizing complex then the class of Gorenstein injective modules is covering.*

Proof. By [3], Theorem 6.9, $\mathcal{GI}$ is closed under direct limits. By Proposition 4, it is covering. □

We extend our results to the category of complexes of R-modules over a two sided noetherian ring R. We will use the notation $\mathcal{GI}(C)$ for the class of Gorenstein injective complexes.

It is known that when R is a left noetherian ring, a complex of left R-modules is Gorenstein injective if and only if each component is a Gorenstein injective R-module ([24], Theorem 8). Using this result we prove:

Proposition 5. *Let R be a two sided noetherian ring. If the class of Gorenstein injective R-modules is closed under filtrations then the class of Gorenstein injective complexes is precovering in $Ch(R)$.*

Proof. Let again κ be an infinite regular cardinal as in Lemma 1, and let $\mathcal{X}$ denote a set of representatives of isomorphism classes of Gorenstein injective modules M such that $|M| \leq \kappa$. Then $\mathcal{GI} = Filt(\mathcal{X})$. Since $\mathcal{GI}$ is closed under filtrations, it follows that each complex of Gorenstein injective modules is filtered by bounded below complexes with components in $\mathcal{X}$ ([26], Proposition 4.3). In particular, the class of complexes of Gorenstein injective modules is deconstructible. By [24] Theorem 8, this is the class of Gorenstein injective complexes. By [26], Theorem (page 2), the class of Gorenstein injective complexes is precovering. □

Proposition 6. *Let R be two sided noetherian. If the class of Gorenstein injective modules is closed under direct limits, then the class of Gorenstein injective complexes is covering in $Ch(R)$.*

Proof. By Proposition 4 the class of Gorenstein injective modules is covering. Also, the class of Gorenstein injective modules is closed under extensions, direct products, and by our assumptions, closed under direct limits. By [10], Theorem 1, the class of complexes of Gorenstein injective modules is covering. By [24], Theorem 8, these are the Gorenstein injective complexes. □

We give a sufficient condition for the class of Gorenstein injective complexes be covering. Since it involves Gorenstein flat modules, we recall the following

Definition 6. *A module N is Gorenstein flat if there is an exact and $Inj \otimes -$ exact sequence $\ldots \to F_1 \to F_0 \to F_{-1} \to F_{-2} \to \ldots$ of flat modules such that $N = Ker(F_0 \to F_{-1})$.*

In [15], Theorem 2, we proved the following result: when the ring R is commutative noetherian and with the property that the character modules of the Gorenstein injective modules are Gorenstein flat, the class of Gorenstein injective modules is closed under direct limits, and so it is covering in $R-Mod$.

By Proposition 6 and [15], Theorem 2, we have :

Theorem 2. *Let R be a commutative noetherian ring. Assume that the character modules of Gorenstein injective modules are Gorenstein flat. Then the class of Gorenstein injective complexes is covering.*

Example 1. *If the ring R is commutative noetherian with a dualizing complex then the class of Gorenstin injective complexes is covering.*

Gorenstein injective envelopes of complexes

In [15] we proved that the class of Gorenstein injective modules is enveloping over a commutative noetherian ring with the property that the character modules of Gorenstein injective modules are Gorenstein flat. In particular this shows the existence of Gorenstein injective envelopes over commutative noetherian rings with dualizing complexes.

We extend this result to the category of complexes. We will denote by $\mathcal{GI}(C)$ the class of Gorenstein injective complexes, and by $\mathcal{GF}(C)$ the class of Gorenstein flat complexes.
We recall the definition of a Gorenstein flat complex. The definition is given ([17]) in terms of the tensor product of complexes.

We recall that if C is a complex of right R-modules and D is a complex of left R-modules then the usual tensor product complex of C and D is the complex of Z-modules $C \otimes^{\cdot} D$ with $(C \otimes^{\cdot} D)_n = \oplus_{t \in Z}(C_t \otimes_R D_{n-t})$ and differentials

$$\delta(x \otimes y) = \delta_t^C(x) \otimes y + (-1)^t x \otimes \delta_{n-t}^D(y)$$

for $x \in C_t$ and $y \in D_{n-t}$.

In [17], García Rozas introduced another tensor product: if C is again a complex of right R-modules and D is a complex of left R-modules then $C \otimes D$ is defined to be $\frac{C \otimes^{\cdot} D}{B(C \otimes^{\cdot} D)}$. Then with the maps

$$\frac{(C \otimes^{\cdot} D)_n}{B_n(C \otimes^{\cdot} D)} \to \frac{(C \otimes^{\cdot} D)_{n-1}}{B_{n-1}(C \otimes^{\cdot} D)}$$

$x \otimes y \to \delta_C(x) \otimes y$, where $x \otimes y$ is used to denote the coset in $\frac{C \otimes^{\cdot} D}{B(C \otimes^{\cdot} D)}$ we get a complex.
This is the tensor product used to define Gorenstein flat complexes.

We recall that a complex F is flat if F is exact and if for each $i \in Z$ the module $Ker(F_i \to F_{i-1})$ is flat ([17], Theorem 4.1.3).

Definition 7. *([17]) A complex X of left R-modules is Gorenstein flat if there exists an exact sequence*

$$\ldots \to F_1 \to F_0 \to F_{-1} \to F_{-2} \to \ldots$$

such that
1) each F_i is a flat complex
2) $C = Ker(F_0 \to F_{-1})$
3) the sequence remains exact when $E \otimes -$ is applied for any injective complex of right R-modules E.

We prove first that if the ring R is noetherian then the class of Gorenstein injective complexes is enveloping if and only if its left orthogonal class, ${}^{\perp}\mathcal{GI}(C)$ is covering.
We start with the following result:

Proposition 7. *Let R be a left noetherian ring. Then $({}^{\perp}\mathcal{GI}(C), \mathcal{GI}(C))$ is a complete hereditary cotorsion pair in the category $Ch(R)$ of complexes of R-modules.*

Proof. By [19], $({}^{\perp}\mathcal{GI}(C), \mathcal{GI}(C))$ is a complete cotorsion pair whenever R is any (left) Noetherian ring.
It remains to show that the pair is hereditary. Let $L \in^{\perp} \mathcal{GI}(C)$ and let G be a Gorenstein injective complex. Then there exists an exact sequence of complexes $0 \to G \to E \to G' \to 0$ with E an injective complex, and with $G' \in \mathcal{GI}(C)$. This gives an exact sequence $0 = Ext^1(L, G') \to Ext^2(L, G) \to Ext^2(L, E) = 0$. Thus $Ext^2(L, G) = 0$. Similarly $Ext^i(L, G) = 0$ for all $i \geq 1$.

□

The following result is proved for modules in [13] (Theorem 1.4). The argument carries to the category of complexes:

Theorem 3. *([13], Theorem 1.4) Let $(\mathcal{L}, \mathcal{C})$ be a hereditary cotorsion pair in $Ch(R)$. Then the following are equivalent:*
(1) $(\mathcal{L}, \mathcal{C})$ is perfect.
(2) Every complex of R-modules has a $\mathcal{C}$ envelope and every $C \in \mathcal{C}$ has an $\mathcal{L}$-cover.
(3) Every complex of R-modules has an $\mathcal{L}$-cover and every $L \in \mathcal{L}$ has a $\mathcal{C}$-envelope.

We use this result to prove the following

Theorem 4. *Let R be a noetherian ring. The following are equivalent:*
(1) The cotorsion pair $({}^{\perp}\mathcal{GI}(C), \mathcal{GI})$ is perfect.
(2) The class ${}^{\perp}\mathcal{GI}(C)$ is covering.
(3) The class $\mathcal{GI}(C)$ is enveloping.

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are immediate from the definition.
(2) $\Rightarrow$ (1) By Theorem 3 it suffices to prove that every complex L in ${}^{\perp}\mathcal{GI}(C)$ has a Gorenstein injective envelope.
Let $L \in^{\perp} \mathcal{GI}(C)$, and let $L \to E$ be an injective envelope. Then we have an exact sequence $0 \to L \to E \to Y \to 0$. Also, by Proposition 7, there exists an exact sequence $0 \to L \to G \to C \to 0$ with $L \to G$ a Gorenstein injective preenvelope and with $C \in^{\perp} \mathcal{GI}(C)$. But then $G \in^{\perp} \mathcal{GI}(C) \cap \mathcal{GI}(C)$, so G is an injective complex. Since $L \to G$ is an injective preenvelope we have $G \simeq E \oplus E'$.
We have a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r] & G \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

This gives an exact sequence $0 \to E \to G \oplus Y \to C \to 0$. Since E is injective we have $G \oplus Y \simeq E \oplus C$ and therefore $C \simeq Y \oplus E'$. It follows that $Y \in^{\perp} \mathcal{GI}(C)$. So the sequence $0 \to L \xrightarrow{i} E \to Y \to 0$ is exact with E Gorenstein injective and with $Y \in^{\perp} \mathcal{GI}(C)$. Thus $L \to E$ is a Gorenstein injective preenvelope. Also, any $u : E \to E$ such that $ui = i$ is an automorphism of E (because $L \to E$ is an injective envelope).
(3) $\Rightarrow$ (1) By Theorem 3, it suffices to show that every complex in $\mathcal{GI}(C)$ has a ${}^{\perp}\mathcal{GI}(C)$ cover.
Let $X \in \mathcal{GI}(C)$. Since $X \in \mathcal{GI}(C)$ there exists an exact sequence $0 \to G \to U \to X \to 0$ with U injective and with G Gorenstein injective. Consider also an exact sequence $0 \to I \to E \to X \to 0$ with $E \to X$ an injective cover. Then $E \in^{\perp} \mathcal{GI}(C)$ and $I \in Inj^{\perp}$.
Since $U \to X$ is an injective precover, we have $U \simeq E \oplus E'$. This gives that $G \simeq I \oplus I'$, so I is Gorenstein injective. Thus the sequence $0 \to I \to E \xrightarrow{\phi} X \to 0$ is exact with $E \in^{\perp} \mathcal{GI}(C)$ and with $I \in \mathcal{GI}(C)$. Then $E \to X$ is a ${}^{\perp}\mathcal{GI}(C)$ precover. Since any $v : E \to E$ such that

$\phi v = \phi$ is an automorphism of E, it follows that $E \xrightarrow{\phi} X$ is a ${}^{\perp}\mathcal{GI}(C)$ cover.

□

For the following result we recall some definitions from [17].
Given two complexes C and D, let $\underline{Hom}(C, D) = Z(\mathcal{H}om(C, D)$. $\underline{Hom}(C, D))$ can be made into a complex with $\underline{Hom}(C, D)_m$ the abelian group of morphisms from C to $D[m]$ and with a boundary operator given by: if $f \in \underline{Hom}(C, D)_m$ then $\delta_m(f) : C \to D[m+1]$ with $\delta_m(f)_n = (-1)^m \delta_D f^n$, for any $n \in \mathbb{Z}$.
The right derived functors of $\underline{Hom}(C, D)$ are complexes denoted $\underline{Ext}^i(C, D)$. $\underline{Ext}^i(C, D) = \ldots \to Ext^i(C, D[n+1]) \to Ext^i(C, D[n]) \to \ldots$, with boundary operator induced by the boundary operator of D.
The right derived functors of the tensor product $-\otimes-$ are denoted by $Tor_i(-, -)$.

We prove that a complex K is in the left orthogonal class of $\mathcal{GI}(C)$ if and only if the complex K^+ is in the right orthogonal class of $\mathcal{GF}(C)$:

Proposition 8. *Let R be a commutative noetherian ring with the property that the character modules of Gorenstein injective modules are Gorenstein flat. Then a complex K is in ${}^\perp\mathcal{GI}(C)$ if and only if $K^+ \in \mathcal{GF}(C)^\perp$.*

Proof. "$\Rightarrow$" Let B be a Gorenstein flat complex. Since R is noetherian this is equivalent to $B_n \in \mathcal{GF}$ for all $n \in \mathbb{Z}$ ([10], Lemma 12). Then B^+ is a complex of Gorenstein injective modules, so $B^+ \in \mathcal{GI}(C)$ (by [24], Theorem 8). So we have $\underline{Ext}^1(K, B^+) = 0$. Then $\underline{Ext}^1(B, K^+) \simeq Tor_1(B, K)^+ \simeq \underline{Ext}^1(K, B^+) = 0$. Thus $Ext^1(B, K^+) = 0$ for any $B \in \mathcal{GF}$. It follows that $K^+ \in \mathcal{GF}(C)^\perp$.

"$\Leftarrow$" Assume that $K^+ \in \mathcal{GF}(C)^\perp$.
Since R is noetherian, by Proposition 7 there exists an exact sequence $0 \to K \to G \to V \to 0$ with G Gorenstein injective and with V in ${}^\perp\mathcal{GI}(C)$. Then we have an exact sequence $0 \to V^+ \to G^+ \to K^+ \to 0$. By the above $V^+ \in \mathcal{GF}(C)^\perp$. By our assumption, $K^+ \in \mathcal{GF}(C)^\perp$. It follows that $G^+ \in \mathcal{GF}(C)^\perp \cap \mathcal{GF}(C)$. But this means that G^+ is a flat complex, and therefore G is an injective complex. So we have an exact sequence $0 \to K \to G \to V \to 0$ with both G and V in ${}^\perp\mathcal{GI}(C)$. Since the pair $({}^\perp\mathcal{GI}(C), \mathcal{GI}(C))$ is hereditary it follows that $K \in^\perp \mathcal{GI}(C)$. □

We can prove now:

Proposition 9. *Let R be commutative noetherian and such that the character modules of Gorenstein injective modules are Gorenstein flat. Then the class ${}^{\perp}\mathcal{GI}(C)$ is closed under pure quotients.*

Proof. Let $0 \to A \to B \to C \to 0$ be a pure exact sequence of complexes with $B \in^{\perp} \mathcal{GI}(C)$. Then the sequence $0 \to C^{+} \to B^{+} \to A^{+} \to 0$ is split exact. So $B^{+} \simeq A^{+} \oplus C^{+}$. By Proposition 8, the complex B^{+} is in $\mathcal{GF}^{\perp}$. It follows that both A^{+} and C^{+} are in $\mathcal{GF}(C)^{\perp}$. By Proposition 8 again, A and C are both in ${}^{\perp}\mathcal{GI}(C)$. □

Theorem 5. *Let R be a commutative noetherian ring such that the character modules of Gorenstein injective modules are Gorenstein flat. Then the class of Gorenstein injective complexes is enveloping in $Ch(R)$.*

Proof. By Theorem 4 it suffices to prove that the class ${}^{\perp}\mathcal{GI}(C)$ is covering.
By Proposition 7 the class ${}^{\perp}\mathcal{GI}(C)$ is precovering, so it is closed under direct sums. Since the direct limit of an inductive family is a pure quotient of the direct sum, by Proposition 9, every direct limit of complexes in ${}^{\perp}\mathcal{GI}(C)$ is still in ${}^{\perp}\mathcal{GI}(C)$. It follows that ${}^{\perp}\mathcal{GI}(C)$ is covering in $Ch(R)$. □

Corollary 4. *If R is a commutative noetherian ring with a dualizing complex, then every complex of R-modules has a Gorenstein injective envelope.*

3. Gorenstein flat and Gorenstein projective precovers for complexes

We recall the following

Definition 8. *A module M is Gorenstein projective if there is an exact and $Hom(-, Proj)$ exact complex $\ldots \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \ldots$ of projective modules such that $M = Ker(P_0 \to P_{-1})$.*

The Gorenstein projective complexes are defined in a similar manner, but working with resolutions of complexes.
We recall that for two complexes X and Y, $Hom(X, Y)$ denotes the group of morphisms of complexes from X to Y.
We also recall that a complex P is projective if the functor $Hom(P, -)$ is exact. Equivalently, P is projective if and only if P is exact and for

each $n \in Z$, $Ker(P_n \to P_{n-1})$ is a projective module. For example, if M is a projective module, then the complex

$$\ldots \to 0 \to M \xrightarrow{Id} M \to 0 \to \ldots$$

is projective. In fact, any projective complex is uniquely up to isomorphism the direct sum of such complexes (one such complex for each $n \in Z$).

By [17], a complex D is called Gorenstein projective if there exists an exact sequence of complexes

$$\ldots \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \ldots$$

such that

1) for each $i \in Z$, P_i is a projective complex
2) $Ker(P_0 \to P_{-1}) = D$
3) the sequence remains exact when $Hom(-, P)$ is applied to it for any projective complex P.

We prove in this section the existence of Gorenstein flat covers of complexes over two sided noetherian rings, and the existence of **special** Gorenstein projective precovers of complexes over commutative noetherian rings of finite Krull dimension. This result generalizes and improves Theorem 4.26 of [25] in two directions: on one side it is established for the category $Ch(R)$ of unbounded complexes, and on the other hand we prove that our Gorenstein precover is *special*. This property has been shown to be crucial in defining the cofibrant and fibrant replacements in (abelian) model category structures on $Ch(R)$ (see [20]). We would like to stress that our methods are necessarily different from those of [25].

We begin by proving the existence of Gorenstein flat precovers and covers over two sided noetherian rings.

Proposition 10. *Let R be a two sided noetherian ring. The class of Gorenstein flat complexes is covering in $Ch(R)$.*

Proof. By [16] Proposition 2.10, the class of Gorenstein flat modules is Kaplansky and closed under direct limits. Then by [26] this class is deconstructible. By [26] Proposition 4.3, the class of complexes of Gorenstein flat modules is deconstructible, so it is precovering. But over a two sided noetherian ring a complex is Gorenstein flat if and

only if it is a complex of Gorenstein flat modules ([10], Lemma 12 and Lemma 13). So the class of Gorenstein flat complexes is precovering. This class of complexes is also closed under direct limits, so it is covering. □

We consider next the question of the existence of Gorenstein projective precovers for complexes.

For modules, Enochs and Jenda showed that when R is a Gorenstein ring the class of Gorenstein projective modules is precovering. Then Jørgensen showed the existence of Gorenstein projective precovers over commutative noetherian rings with dualizing complexes. Recently, Murfet and Salarian extended his result to commutative noetherian rings of finite Krull dimension.

Their goal in [25] was to introduce a triangulated category of totally acyclic complexes of flat modules, which plays the role of $K_{tac}(ProjR)$ for any noetherian ring R (in fact they work in a more general setting, that of complexes of flat sheaves over noetherian schemes).

To accomplish this they started with a construction developed by Neeman, who defined $N(Flat)$ as the Verdier quotient $\frac{K(Flat)}{K_{pac}(Flat)}$, with $K(Flat)$ the homotopy category of complexes of flat modules, and $K_{pac}(Flat)$ the full subcategory of pure acyclic complexes in $K(Flat)$ (it is known that a complex of flat modules is pure acyclic if and only if it is a flat complex in the sense of García Rozas' definition from [21]). Then they considered the full subcategory of $N(Flat)$, $N_{tac}(Flat)$, of N-totally acyclic complexes of flat modules (i.e. exact and $Inj \otimes -$ exact complexes of flat modules). Their results in [25] indicate that this is the correct triangulated category one can use in order to generalize aspects of Gorenstein homological algebra to schemes.

We show that when R is noetherian the class of N-totally acyclic complexes of flat modules is precovering.

In the following we use $\widetilde{GorFlat}$ to denote the class of exact complexes F with $Z_n(F) \in \mathcal{GF}$ for each n. Since the class of Gorenstein flat modules is Kaplansky and is also closed under direct limits, extensions and retracts, the class $\widetilde{GorFlat}$ is covering in $Ch(R)$ (by [20, Theorem 4.12], or see also [16, Corollary 2.11] and [9, Corollary 3.1]). By [20], its right orthogonal class, $\widetilde{GorFlat}^{\perp}$, consists of the complexes X with each $X_n \in \mathcal{GF}^{\perp}$ and such that for any $G \in \widetilde{GorFlat}$, every $u \in Hom(G, X)$ is homotopic to zero.

Proposition 11. *Let R be a noetherian ring. Then the class of N-totally acyclic complexes of flat modules is precovering in $Ch(R)$.*

Proof. - Let P be a complex of flat R-modules. Since the class of $\widetilde{GorFlat}$ complexes is covering, there is an exact sequence

$$0 \to K \to F \to P \to 0$$

with $F \in \widetilde{GorFlat}$ and with $K \in \widetilde{GorFlat}^{\perp}$. In particular, each module K_n is in $\mathcal{GF}^{\perp}$.
For each n we have an exact sequence

$$0 \to K_n \to F_n \to P_n \to 0$$

Since P_n is flat and $K_n \in \mathcal{GF}^{\perp}$, the sequence is split exact. So K_n is a direct summand of F_n, so it is Gorenstein flat. But then $K_n \in \mathcal{GF} \bigcap \mathcal{GF}^{\perp}$ gives that K_n is flat for each n. Therefore F_n is flat for each n. So the complex F is N-totally acyclic. Also, for each N-totally acyclic complex D we have that D is in $\widetilde{GorFlat}$, so $Ext^1(D, K) = 0$.

- Let X be any complex of R-modules. Since the class of exact complexes of flat modules, $dw(Flat) \bigcap \mathcal{E}$, is precovering ([2], example 2), there is an exact sequence

$$0 \to H \to P \to X \to 0$$

with P exact complex of flat modules and with H in $(dw(Flat) \bigcap \mathcal{E})^{\perp}$.
By the above there is an exact sequence

$$0 \to K \to F \to P \to 0$$

with F an N-totally acyclic complex of flat modules and $K \in N_{tac}(Flat)^{\perp}$.
We form the commutative diagram

$$\begin{tikzcd}
 & K \arrow[r, equal] \arrow[d] & K \arrow[d] & & \\
0 \arrow[r] & M \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H \arrow[r] & P \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

So we have an exact sequence

$$0 \to M \to F \to X \to 0$$

with F N-totally acyclic complex of flat modules. Both K and H are in $N_{tac}(Flat)^{\perp}$, so M also satisfies $Ext^1(D, M) = 0$ for any N-totally acyclic complex D. $\square$

We recall that over a commutative noetherian ring of finite Krull dimension d every Gorenstein flat module M has finite Gorenstein projective dimension, and $G.p.d_R(M) \leq d$.

We prove now the existence of **special** Gorenstein projective precovers in $Ch(R)$ over a commutative noetherian ring R of finite Krull dimension.

The proof uses the fact that over such a ring R, a complex is Gorenstein projective if and only if it is a complex of Gorenstein projective R-modules ([10], Theorem 3).

Proposition 12. *If R is commutative noetherian of finite Krull dimension, then every complex X of R-modules has a special Gorenstein projective precover.*

Proof. Let $dim\ R = d$.
- We show first that every Gorenstein flat complex G has a special Gorenstein projective precover.

Let

$$0 \to \overline{G} \to P_{d-1} \to \ldots \to P_0 \to G \to 0$$

be a partial projective resolution of G. Then for each j we have an exact sequence of modules

$$0 \to \overline{G}_j \to P_{d-1,j} \to \ldots \to P_{0,j} \to G_j \to 0$$

Since $Gpd\ G_j \leq d$ it follows that each $\overline{G}_j$ is Gorenstein projective. Thus $\overline{G}$ is a Gorenstein projective complex (by [10], Theorem 3). So $\overline{G}$ has an exact and $Hom(-, Proj)$ exact complex of projective complexes

$$0 \to \overline{G} \to T_{d-1} \to \ldots \to T_0 \to \ldots$$

Let $T = Ker(T_{-1} \to T_{-2})$. Then T is a Gorenstein projective complex, and we have a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \overline{G} \arrow[r] \arrow[d, equal] & T_{d-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & T_1 \arrow[r] \arrow[d] & T_0 \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overline{G} \arrow[r] & P_{d-1} \arrow[r] & \cdots \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & G \arrow[r] & 0
\end{tikzcd}$$

Therefore we have an exact sequence:

$$0 \to T_{d-1} \to P_{d-1} \oplus T_{d-2} \to \ldots \to P_1 \oplus T_0 \to P_0 \oplus T \xrightarrow{\delta} G \to 0$$

Let $V = Ker\delta$. Then V has finite projective dimension, so $Ext^1(W, V) = 0$ for any Gorenstein projective complex W.
We have an exact sequence $0 \to V \to P_0 \oplus T \to G \to 0$ with $P_0 \oplus T$ Gorenstein projective and with V of finite projective dimension. Thus $P_0 \oplus T \to G$ is a special Gorenstein projective precover.

- We prove now that every complex X has a special Gorenstein projective precover.

Let X be any complex of R-modules. By Proposition 10, X has an exact sequence

$$0 \to Y \to G \to X \to 0$$

with G Gorenstein flat and with $Ext^1(U, Y) = 0$ for any Gorenstein flat complex U.

By the above, there is an exact sequence

$$0 \to L \to P \to G \to 0$$

with P Gorenstein projective and with L complex of finite projective dimension.
Form the pullback diagram

$$\begin{tikzcd}
 & L \arrow[r, equal] \arrow[d] & L \arrow[d] & & \\
0 \arrow[r] & M \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r] & G \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

Since $L \in GorProj^{\perp}$ and $Y \in GorFlat^{\perp}$ and the sequence $0 \to L \to M \to Y \to 0$ is exact, it follows that $M \in GorProj^{\perp}$.
So $0 \to M \to P \to X \to 0$ is exact with P Gorenstein projective and with $M \in GorProj^{\perp}$

□

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